

RIEMANNIAN MANIFOLDS WITH POSITIVE SECTIONAL CURVATURE

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Of special interest in the history of Riemannian geometry have been manifolds with positive sectional curvature. In these notes we want to give a survey of this subject and some recent developments. We start with some historical developments.

1. HISTORY AND OBSTRUCTIONS

It is fair to say that Riemannian geometry started with Gauss's famous "Disquisitiones generales" from 1827 in which one finds a rigorous discussion of what we now call the Gauss curvature of a surface. Much has been written about the importance and influence of this paper, see in particular the article [Do] by P.Dombrowski for a careful discussion of its contents and influence during that time. Here we only make a few comments. Curvature of surfaces in 3-space had been studied previously by a number of authors and was defined as the product of the principal curvatures. But Gauss was the first to make the surprising discovery that this curvature only depends on the intrinsic metric and not on the embedding. Here one finds for example the formula for the metric in the form $ds^2 = dr^2 + f(r, \theta)^2 d\theta^2$. Gauss showed that every metric on a surface has this form in "normal" coordinates and that it has curvature $K = -f_{rr}/f$. In fact one can take it as the definition of the Gauss curvature and proves Gauss's famous "Theorema Egregium" that the curvature is an intrinsic invariant and does not depend on the embedding in \mathbb{R}^3 . He also proved a local version of what we nowadays call the Gauss-Bonnet theorem (it is not clear what Bonnet's contribution was to this result), which states that in a geodesic triangle Δ with angles α, β, γ the Gauss curvature measures the angle "defect":

$$\int_{\Delta} K dvol = \alpha + \beta + \gamma - \pi$$

Nowadays the Gauss Bonnet theorem also goes under its global formulation for a compact surface:

$$\int_M K dvol = 2\pi\chi(M)$$

where $\chi(M)$ is the Euler characteristic. This follows from the defect formula by using a triangulation, but it is actually not found in any of Gauss's papers. Of course no rigorous definition of a manifold or of the Euler characteristic existed at the time. Maybe the first time the above formulation can be found is in Blaschke's famous book "Vorlesungen ueber Differential Geometrie" from 1921 [Bl] (although it is already discussed in a paper by Boy in 1903 [Bo]).

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In any case, the formula implies that a compact surface with positive curvature must be the 2-sphere, or the real projective plane. This is of course the beginning of the topic of these lectures on positive curvature.

The next big step was made by Riemann in his famous Habilitation from 1854, eight months before Gauss's death). He started what we now aptly call Riemannian geometry (in dimension bigger than 2) by giving intrinsic definitions of what is now called sectional curvature (we will use \sec for this notion instead of the more common one K). For each 2-plane $\sigma \subset T_p M$ one associates the sectional curvature, which we denote by $\sec(\sigma)$. This can be defined for example as the Gauss curvature of the 2-dimensional surface spanned by going along geodesics in the direction of σ (this was in fact one of Riemann's definitions). Here one also finds for the first time an explicit formula for a space of constant curvature c :

$$ds^2 = \frac{dx_1^2 + \dots dx_n^2}{1 + \frac{c}{4}(x_1^2 + \dots x_n^2)}$$

including in particular the important case of the hyperbolic plane $c = -1$.

For our story, the next important development was Clifford's discovery in 1873 of the Clifford torus $\mathbb{S}^1(1) \times \mathbb{S}^1(1) \subset \mathbb{S}^3(\sqrt{2}) \subset \mathbb{R}^4$, which to his surprise has intrinsic curvature 0 (After all something that looks like a plane has to extend to infinity). This motivated Klein to formulate his famous Clifford-Klein space form problem, which in one formulation asks to classify surfaces of constant curvature. This has a painful history (after all, one needs a good definition of completeness, a concept of a global surface and some understanding of covering space theory). In a beautiful paper by H. Hopf from 1926 [Ho] he gave us our present definition of completeness and solves the classification problem. It is amusing to note that Hopf points out that the many previous papers on the subject, especially by Killing in 1891-1893, the authors did not realize that the Moebius band has a flat metric.

For us the next development is of course the Bonnet-Myers theorem, which holds more generally for the positivity of an average of the sectional curvatures [My2]:

THEOREM (Bonnet-Myers). *If M has a complete metric with $\text{Ric} \geq 1$ then the diameter is at most π , and the fundamental group is finite.*

This theorem also has an interesting history. Bonnet in 1855 only showed that the "extrinsic" diameter in 3-space has length at most π . The difficulty to obtain an intrinsic proof in higher dimensions was partially due to the fact that one needs a good formula for the second variation, which surprisingly took a long time to develop. Noteworthy are papers by Synge from 1925 [Sy1] (who was the first one to show that a geodesic of length $> \pi$ cannot be shortest by a second variation argument), Hopf-Rinow from 1931 [HR] (where they proved any two points can be joined by a minimal geodesic) Schoenberg from 1932 [Sch], Myers from 1935 [My1] (here one finds for the first time the conclusion that $\pi_1(M)$ is finite) and Synge [Sy2] from 1935 as well. There was a fierce competition between Myers and Synge for priorities ([My1] and [Sy2] appeared in the same issue of Duke Math J. and in Myers paper one finds the mysterious footnote "Received by the Editors of the Annals of Mathematics, February 27, 1934, accepted by them, and later transferred to this journal"). Schoenberg's paper contains the formula for second variation that one now finds in books, and Synge's papers the usual proof in the case of sectional curvature. In 1941

Myers used this proof and summed over an orthonormal basis. Thus it would be fairer to call it the Bonnet-Synge-Myers theorem. Nevertheless, Myers paper created a lot of excitement at the time due to the importance of Ricci curvature in general relativity.

Important for our story is another paper by Synge from 1936 [Sy3] where he proved:

THEOREM (Synge). *If M is a compact manifold with positive sectional curvature, then $\pi_1(M)$ is 0 or \mathbb{Z}_2 if n is even, and M is orientable if n is odd.*

In particular, $\mathbb{RP}^n \times \mathbb{RP}^n$ does not admit a metric with positive curvature. I can also recommend reading Preissman's paper from 1936 [Pr] on negative curvature, still very readable for today's audience.

The surprising fact is that the above two theorems are the only known obstructions that deal with positive curvature only. There are a number theorems that give obstructions to non-negative curvature. On the other hand, one expects that the class of manifolds admitting positive curvature is much smaller than the class admitting non-negative curvature (and this is born out in known examples). Since this is not the purpose of the present notes, we just summarize them:

- (Gromov) If M^n is a compact manifold with $\sec \geq 0$, then there exists a universal constant $c(n)$ such that $b_i(M^n, F) \leq c(n)$ for all i and any field of coefficients F . Furthermore, the fundamental group has a generating set with at most $c(n)$ elements.
- (Cheeger-Gromoll) If M^n is a compact manifold that admits a metric with non-negative sectional curvature, then there exists an abelian subgroup of $\pi_1(M^n)$ with finite index.
- (Lichnerowicz-Hitchin) The obstructions to positive scalar curvature imply that a compact spin manifold with $\hat{A}(M) \neq 0$ or $\alpha(M) \neq 0$ does not admit a metric with non-negative sectional curvature, unless it is flat. In particular there exist exotic spheres, e.g. in dimension 9, which do not admit positive curvature.
- (Cheeger-Gromoll) If M^n is a non-compact manifold with a complete metric with $\sec \geq 0$, then there exists a totally geodesic compact submanifold S^k , called the soul, such that M^n is diffeomorphic to the normal bundle of S^k .

If we allow ourselves to add an upper as well as a lower bound on the sectional curvature it is convenient to introduce what is called the *pinching constant* which is defined as $\delta = \min \sec / \max \sec$. One then has the following recognition and finiteness theorems:

- (Berger-Klingenberg, Brendle-Schoen) If M^n is a compact manifold with $\delta \geq \frac{1}{4}$, then M is either diffeomorphic to a space form \mathbb{S}^n/Γ or isometric to \mathbb{CP}^n , \mathbb{HP}^n or CaP^2 with their standard Fubini metric.
- (Cheeger) Given a positive constant ϵ , there are only finitely many diffeomorphism types of compact simply connected manifolds M^{2n} with $\delta \geq \epsilon$.
- (Fang-Rong, Petrunin-Tuschmann) Given a positive constant ϵ , there are only finitely many diffeomorphism types of compact manifolds M^{2n+1} with $\pi_1(M) = \pi_2(M) = 0$ and $\delta \geq \epsilon$.

Since our emphasis is positive curvature, we will not discuss other results, except in passing, about non-negative curvature in this survey. We finally mention some conjectures.

- (Hopf) There exists no metric with positive sectional curvature on $\mathbb{S}^2 \times \mathbb{S}^2$. More generally, there are no positively curved metrics on the product of two compact manifolds, or on a symmetric space of rank at least two.
- (Hopf) A compact manifold with $\sec \geq 0$ has non-negative Euler characteristic. An even dimensional manifold with positive curvature has positive Euler characteristic.
- (Bott-Grove-Halperin) A compact simply connected manifold M with $\sec \geq 0$ is *elliptic*, i.e., the sequence of Betti numbers of the loop space of M grows at most polynomially for every field of coefficients.

The latter conjecture, and its many consequences, were discussed in the literature for the first time in [GH]. It is usually formulated for rational coefficients, where it is equivalent to the condition that only finitely many homotopy groups are non-zero (called rationally elliptic). One can thus apply rational homotopy theory to obtain many consequences. E.g., it implies, under the assumption of non-negative curvature, that $b_i(M^n, F) \leq 2^n$ and that the Euler characteristic is non-negative (Hopf conjecture), and positive in even dimensions iff all odd Betti numbers vanish. The above more geometric formulation, which one should call elliptic, is a natural generalization. If $n = 4$, rational homotopy theory implies that M , if compact and simply connected, is diffeomorphic to one of the known examples of non-negative curvature, i.e. $\mathbb{S}^4, \mathbb{CP}^2, \mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$. In [PP] it was shown that a compact simply connected elliptic 5-manifold is diffeomorphic to one of the known examples with non-negative curvature, i.e., one of $\mathbb{S}^5, \mathrm{SU}(3)/\mathrm{SO}(3), \mathbb{S}^3 \times \mathbb{S}^2$ or the non-trivial \mathbb{S}^3 bundle over \mathbb{S}^2 . In both cases, no curvature assumption is necessary.

Of course, one should also mention Hamilton's theorem which states that a 3 manifold with positive curvature is diffeomorphic to a space form \mathbb{S}^4/Γ . Thus in dimension 2 and 3, manifolds with positive curvature are classified.

We formulate some other natural conjectures:

- A compact simply connected 4 manifold with positive curvature is diffeomorphic to \mathbb{S}^4 or \mathbb{CP}^2 .
- A compact simply connected 5 manifold with positive curvature is diffeomorphic to \mathbb{S}^5 .
- (Klingenberg-Sakai) There are only finitely many diffeomorphism classes of positively curved manifolds in a given homotopy type.
- There are only finitely many diffeomorphism classes of positively curved manifolds in even dimensions, and all odd Betti numbers are 0.
- In odd dimension, there are only finitely many 2-connected manifolds with positive curvature.

The last 2 finiteness conjectures are probably too optimistic, but one should at least expect an upper bound on the Betti numbers, e.g. at most 2 in dimension 6.

2. COMPACT EXAMPLES OF POSITIVE CURVATURE

Homogeneous spaces which admit a homogeneous metric with positive curvature have been classified by Wallach in even dimensions ([Wa]) and by Bérard-Bergery in odd dimensions ([BB]). We now describe these examples, due to Berger, Wallach and Aloff-Wallach

[Be, Wa, AW], as well as the biquotient examples [E1, E2, Ba]. In most cases we will also mention that they admit natural fibrations, a topic we will cover in Section 4.

1) The god given basic examples of positive curvature are the rank one symmetric spaces \mathbb{S}^n , \mathbb{CP}^n , \mathbb{HP}^n or \mathbb{CaP}^2 . (We do not know where in the literature it is first discussed that \mathbb{CaP}^2 carries a metric with positive curvature which is 1/4 pinched). They admit the well known homogeneous Hopf fibrations. Recall that a homogeneous fibration is of the form $K/H \rightarrow G/H \rightarrow G/K$ obtained from inclusions $H \subset K \subset G$.

$$\mathbb{S}^1 \rightarrow \mathbb{S}^{2n+1} \rightarrow \mathbb{CP}^n \text{ obtained from } \mathrm{SU}(n) \subset \mathrm{U}(n) \subset \mathrm{SU}(n+1),$$

$$\mathbb{S}^3 \rightarrow \mathbb{S}^{4n+3} \rightarrow \mathbb{HP}^n \text{ obtained from } \mathrm{Sp}(n) \subset \mathrm{Sp}(n)\mathrm{Sp}(1) \subset \mathrm{Sp}(n+1),$$

$$\mathbb{S}^2 \rightarrow \mathbb{CP}^{2n+1} \rightarrow \mathbb{HP}^n \text{ obtained from } \mathrm{Sp}(n)\mathrm{U}(1) \subset \mathrm{Sp}(n)\mathrm{Sp}(1) \subset \mathrm{Sp}(n+1).$$

$$\mathbb{S}^7 \rightarrow \mathbb{S}^{15} \rightarrow \mathbb{S}^8 \text{ coming from } \mathrm{Spin}(7) \subset \mathrm{Spin}(8) \subset \mathrm{Spin}(9).$$

2) The homogeneous flag manifolds due to Wallach: $W^6 = \mathrm{SU}(3)/\mathrm{T}^2$, $W^{12} = \mathrm{Sp}(3)/\mathrm{Sp}(1)^3$ and $W^{24} = \mathrm{F}_4/\mathrm{Spin}(8)$. They are the total space of the following homogeneous fibrations:

$$\mathbb{S}^2 \rightarrow \mathrm{SU}(3)/\mathrm{T}^2 \rightarrow \mathbb{CP}^2,$$

$$\mathbb{S}^4 \rightarrow \mathrm{Sp}(3)/\mathrm{Sp}(1)^3 \rightarrow \mathbb{HP}^2,$$

$$\mathbb{S}^8 \rightarrow \mathrm{F}_4/\mathrm{Spin}(8) \rightarrow \mathbb{CaP}^2.$$

3) The Berger space $B^{13} = \mathrm{SU}(5)/\mathrm{Sp}(2) \cdot \mathrm{S}^1$ admits a fibration

$$\mathbb{RP}^5 \rightarrow \mathrm{SU}(5)/\mathrm{Sp}(2) \cdot \mathrm{S}^1 \rightarrow \mathbb{CP}^4,$$

coming from the inclusions $\mathrm{Sp}(2) \cdot \mathrm{S}^1 \subset \mathrm{U}(4) \subset \mathrm{SU}(5)$. Here $\mathrm{Sp}(2) \subset \mathrm{SU}(4)$ is the usual embedding and S^1 is the center of $\mathrm{U}(4)$. Furthermore, the fiber is $\mathrm{U}(4)/\mathrm{Sp}(2) \cdot \mathrm{S}^1 = \mathrm{SU}(4)/\mathrm{Sp}(2) \cdot \mathbb{Z}_2 = \mathrm{SO}(6)/\mathrm{O}(5) = \mathbb{RP}^5$.

4) The Aloff-Wallach spaces $W_{p,q}^7 = \mathrm{SU}(3)/\mathrm{diag}(z^p, z^q, \bar{z}^{p+q})$, $\mathrm{gcd}(p, q) = 1$. By interchanging coordinates we can assume $p \geq q \geq 0$. They have positive curvature, unless $(p, q) = (1, 0)$. They also admit interesting fibrations

$$\mathbb{S}^3/\mathbb{Z}_{p+q} \rightarrow W_{p,q} \rightarrow \mathrm{SU}(3)/\mathrm{U}(2),$$

coming from the inclusions $\mathrm{diag}(z^p, z^q, \bar{z}^{p+q}) \subset \mathrm{U}(2) \subset \mathrm{SU}(3)$. Hence, as long as $p + q \neq 0$ (or $q \neq 0$ in the above notation), the fiber is the lens space $\mathrm{U}(2)/\mathrm{diag}(z^p, z^q) = \mathrm{SU}(2)/\mathrm{diag}(z^p, z^q)$ with $z^{p+q} = 1$. In the special case of $p = q = 1$, we obtain a principal $\mathrm{SO}(3)$ bundle.

Another fibration is of the form

$$\mathbb{S}^1 \rightarrow W_{p,q} \rightarrow \mathrm{SU}(3)/\mathrm{T}^2,$$

coming from the inclusions $\mathrm{diag}(z^p, z^q, \bar{z}^{p+q}) \subset \mathrm{T}^2 \subset \mathrm{SU}(3)$.

5) The Berger space: $B^7 = \mathrm{SO}(5)/\mathrm{SO}(3)$. To describe the embedding $\mathrm{SO}(3) \subset \mathrm{SO}(5)$, we recall that $\mathrm{SO}(3)$ acts orthogonally via conjugation on the set of 3×3 symmetric traceless matrices. This space is special since $\mathrm{SO}(3)$ is maximal in $\mathrm{SO}(5)$ and hence does not admit a homogeneous fibration. On the other hand, in [GKS] it was shown that the manifold is diffeomorphic to an \mathbb{S}^3 bundle over \mathbb{S}^4 . It is also what is called isotropy irreducible, i.e., the isotropy action of H on the tangent space is irreducible. This implies that there is only one $\mathrm{SO}(5)$ invariant metric up to scaling.

Thus all of these examples in 2)- 5) are the total space of a fibration. This property will be interesting to us in Section 4.

A natural generalization of homogeneous spaces are so called biquotients, discussed for the first time in [GM]. For this, let G/H be a homogeneous space and $K \subset G$ a subgroup. Then K acts on G/H on the left, and in some cases the action is free, in which case the manifold $K \backslash G/H$ is a biquotient. An equivalent formulation is as follows: Take a subgroup $U \subset G \times G$ and let U act on the left and right $(u_1, u_2) * g = u_1 g u_2^{-1}$. The action is free, if for any $(u_1, u_2) \in U$ the element u_1 is not conjugate to u_2 unless $u_1 = u_2$ lies in the center of G . We denote the quotient by $G//U$. The biinvariant metric on G (or $G \times G$) induces a metric on $G//U$ with non-negative sectional curvature. In some cases, this can be deformed (via a Cheeger deformation) into one with positive curvature. We now describe these biquotient examples, due to Eschenburg and Bazaikin, explicitly.

6) There is an analogue of the 6-dimensional flag manifold which is a biquotient of $\mathrm{SU}(3)$ under an action of $T^2 = \{(z, w) \mid z, w \in \mathbb{C}, |z| = |w| = 1\}$. It is given by:

$$E^6 = \mathrm{SU}(3)//T^2 = \mathrm{diag}(z, w, zw) \backslash \mathrm{SU}(3) / \mathrm{diag}(1, 1, z^2 w^2)^{-1}.$$

The action by T^2 is clearly free. In order to show that this manifold is not diffeomorphic to the homogeneous flag W^6 , one needs to compute the cohomology with integer coefficients. The cohomology groups are the same for both manifolds, but the ring structure is different ([E2]). The examples W^6 and E^6 , which have $b_2 = 2$, as well as \mathbb{S}^6 and \mathbb{CP}^3 , are the only known examples of positive curvature. It is thus a natural question whether positive curvature in dimension 6 implies that the Betti numbers satisfy $b_1 = b_3 = b_5 = 0$ and $b_2 = b_4 \leq 2$.

The inhomogeneous flag also admits a fibration of a (different) sphere bundle similar to the flag manifold:

$$\mathbb{S}^2 \rightarrow \mathrm{SU}(3)//T^2 \rightarrow \mathbb{CP}^2$$

7) We now describe the 7-dimensional family of Eschenburg spaces $E_{k,l}$, which can be considered as a generalization of the Aloff Wallach spaces. Let $k := (k_1, k_2, k_3)$ and $l := (l_1, l_2, l_3) \in \mathbb{Z}^3$ be two triples of integers with $\sum k_i = \sum l_i$. We can then define a two-sided action of $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ on $\mathrm{SU}(3)$ whose quotient we denote by $E_{k,l}$:

$$E_{k,l} = \mathrm{SU}(3)//S^1 = \mathrm{diag}(z^{k_1}, z^{k_2}, z^{k_3}) \backslash \mathrm{SU}(3) / \mathrm{diag}(z^{l_1}, z^{l_2}, z^{l_3})^{-1}.$$

The action is free if and only if $\mathrm{diag}(z^{k_1}, z^{k_2}, z^{k_3})$ is not conjugate to $\mathrm{diag}(z^{l_1}, z^{l_2}, z^{l_3})$, i.e.

$$\gcd(k_1 - l_i, k_2 - l_j) = 1, \text{ for all } i \neq j, i, j \in \{1, 2, 3\}.$$

Eschenburg showed that $E_{k,l}$ has positive curvature if

$$k_i \notin [\min(l_1, l_2, l_3), \max(l_1, l_2, l_3)].$$

Among the biquotients $E_{k,l}$ there are two interesting subfamilies. $E_p = E_{k,l}$ with $k = (1, 1, p)$ and $l = (1, 1, p + 2)$ has positive curvature when $p > 0$. It admits a large group acting by isometries. Indeed, $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$ acting on $\mathrm{SU}(3)$ on the left and on the right, acts by isometries in the Eschenburg metric and commutes with the S^1 action. Thus it acts by isometries on E_p and one easily sees that E_p/G is one dimensional, i.e., E_p is cohomogeneity one. A second family consists of the cohomogeneity two Eschenburg spaces $E_{a,b,c} = E_{k,l}$ with $k = (a, b, c)$ and $l = (1, 1, a + b + c)$. Here $c = -(a + b)$ is the subfamily of Aloff-Wallach spaces. The action is free iff a, b, c are pairwise relatively prime and the Eschenburg metric has positive curvature iff, up to permutations, $a \geq b \geq c > 0$ or $a \geq b > 0, c < -a$. For these spaces $G = \mathrm{U}(2)$ acts by isometries on the right and $E_{a,b,c}/G$ is two dimensional. For a general Eschenburg space $G = \mathrm{T}^3$ acts by isometries and $E_{k,l}/G$ is four dimensional. In [GSZ] it was shown that these groups G are indeed the id component of the full isometry group of a positively curved Eschenburg space, unless it is an Aloff-Wallach space.

There are again natural fibrations. In the case of $E_{a,b,a+b}$ with $a \geq b > 0$, the circle fibrations:

$$\mathrm{S}^1 \rightarrow E_{a,b,a+b} \rightarrow \mathrm{SU}(3) // \mathrm{T}^2,$$

and the lens space fibrations:

$$\mathrm{S}^3 / \mathbb{Z}_{a+b} \rightarrow E_{a,b,a+b} \rightarrow \mathbb{CP}^2$$

which, in the case of $a = b = 1$, gives a second $\mathrm{SO}(3)$ principal bundle over \mathbb{CP}^2 .

The cohomogeneity two Eschenburg spaces admit orbifold fibrations, which will also be of interest to us in Section 4.

$$F \rightarrow E_{a,b,c} \rightarrow \mathbb{CP}^2[a + b, a + c, b + c],$$

where the fiber F is \mathbb{RP}^3 if all a, b, c 's are odd, and $F = \mathrm{S}^3$ otherwise. Here the base is a 2-dimensional weighted complex projective space. A general Eschenburg space is the total space of an orbifold circle bundle [FZ1].

8) We finally have the 13-dimensional Bazaikin spaces B_q , which can be considered as a generalization of the Berger space B^{13} . Let $q = (q_0, \dots, q_5)$ be a 6-tuple of integers with $\sum q_i = 0$ and define

$$B_q = \mathrm{diag}(z^{q_1}, \dots, z^{q_5}) \backslash \mathrm{SU}(5) / \mathrm{diag}(z^{-q_0}, A)^{-1},$$

where $A \in \mathrm{Sp}(2) \subset \mathrm{SU}(4) \subset \mathrm{SU}(5)$. Here we follow the treatment in [Zi1] of Bazaikin's work [Ba] (see also [EKS]). First, one easily shows that the action of $\mathrm{Sp}(2) \cdot \mathrm{S}^1$ is free if and only if

$$\text{all } q_i \text{'s are odd and } \gcd(q_{\sigma(1)} + q_{\sigma(2)}, q_{\sigma(3)} + q_{\sigma(4)}) = 2,$$

for all permutations $\sigma \in S_5$. On $\mathrm{SU}(5)$ we choose an Eschenburg metric by scaling the biinvariant metric on $\mathrm{SU}(5)$ in the direction of $\mathrm{U}(4) \subset \mathrm{SU}(5)$. The right action of $\mathrm{Sp}(2) \cdot \mathrm{S}^1$

is then by isometries. Repeating the same arguments as in the previous case, one shows that the induced metric on $SU(5)//Sp(2) \cdot S^1$ satisfies

$$\sec > 0 \quad \text{if and only if} \quad q_i + q_j > 0 \quad (\text{or} < 0) \quad \text{for all } i < j.$$

The special case of $q = (1, 1, 1, 1, 1)$ is the homogeneous Berger space. One again has a one parameter subfamily that is cohomogeneity one, given by $B_p = B_{(1,1,1,1,2p-1)}$ since $U(4)$ acting on the left induces an isometric action on the quotient. It has positive curvature when $p \geq 1$.

There is another equivalent description of the Bazaikin spaces given by

$$B_q = \text{diag}(z^{q_1}, \dots, z^{q_6}) \backslash SU(6)/Sp(3)$$

with $\sum q_i = 0$.

For these manifolds one has natural fibrations obtained from both descriptions, given by

$$S^1 \rightarrow SU(6)/Sp(3) \rightarrow B_q,$$

and

$$S^5 \rightarrow SU(6)/Sp(3) \rightarrow S^9.$$

But in this case B_q is not the total space of a fibration, unless it is homogeneous. On the other hand, if we allow orbifold fibrations they all admit one:

$$\mathbb{RP}^5 \rightarrow B_q \rightarrow \mathbb{CP}^4[q_0 + q_1, q_0 + q_2, \dots, q_0 + q_6].$$

Unlike in the homogeneous case, there is no general classification of positively curved biquotients, except in the following cases. We call a metric on $G//H$ torus invariant if it is induced by a left invariant metric on G which is also right invariant under the action of a maximal torus. The main theorem in [E2] states that an even dimensional biquotient $G//H$ with G simple and which admits a positively curved torus invariant metric is diffeomorphic to a rank one symmetric space or the biquotient $SU(3)//T^2$. In the odd dimensional case he shows that $G//H$ with a positively curved torus invariant metric and G of rank 2 is either diffeomorphic to a homogeneous space or a positively curved Eschenburg space.

There are only two more examples which are not homogeneous or biquotients. One is a 7-dimensional exotic sphere due to Petersen-Wilhem [PW] (although the rather delicate calculations have not yet been verified). The method is via deforming a natural metric of non-negative curvature on a biquotient description of the exotic sphere to positive curvature. Deforming non-negative curvature to positive curvature is an important problem, and not yet well understood.

The second example is due to Grove-Verdiani-Ziller [GVZ], and independently O.Dearriscott [De], and will be discussed in Section 4. It arises as the total space of an orbifold fibration.

It is also interesting to examine the topology of the known examples. In [KS] it was shown that there exist pairs of positively curved Aloff Wallach spaces which are homeomorphic but not diffeomorphic. This turns out to happen more frequently for the Eschenburg spaces [CEZ]. For such pairs M, M' one knows that $M = M' \# \Sigma^7$, for some exotic sphere Σ^7 . It is not hard to check that among the examples in [CEZ], every exotic 7-sphere can occur as a factor Σ^7 , whereas this does not seem to be the case for Aloff Wallach spaces.

On the other hand, [FZ2] provides evidence that positively curved Bazaikin spaces are homeomorphically distinct.

3. POSITIVE CURVATURE WITH SYMMETRY

As we saw in Section 1, not much is known as far as general obstructions to positive curvature is concerned. A very successful program was suggested by K.Grove, motivated by the Kleiner-Hsiang theorem below [HK], that one should examine positive curvature under the additional assumption of a large symmetry group.

THEOREM (Kleiner-Hsiang). *If M is a compact simply connected 4-manifold on which a circle acts by isometries, then M is homeomorphic to \mathbb{S}^4 or \mathbb{S}^2 .*

Topological results on circle actions imply that they are diffeomorphic, and a recent result by Grove-Wilking shows that the \mathbb{S}^1 action is linear. Thus a counter example to the Hopf conjecture would have to have a finite isometry group.

In higher dimensions, one obtains obstructions assuming that a torus of large dimension acts [GS].

THEOREM (Grove-Searle). *If M^n is a compact simply connected manifold with positive curvature on which a torus \mathbb{T}^s acts by isometries, then $s \leq n/2$ in even dimensions, and $s \leq (n+1)/2$ in odd dimensions. Equality holds iff M is diffeomorphic to \mathbb{S}^n or \mathbb{CP}^n .*

See the article in this volume where it is shown that the torus action is linear as well. Great strides were made by B.Wilking [Wi1, Wi2] who showed

THEOREM (Wilking). *Let M^n be a compact simply connected manifold with positive curvature.*

- (a) *If $n \neq 7$ and \mathbb{T}^s acts by isometries with $s \geq \frac{n+1}{4}$, then M is homotopy equivalent to a rank one symmetric space.*
- (b) *If $0 < \dim M/G < \sqrt{n/18} - 1$, then M is homotopy equivalent to a rank one symmetric space.*
- (c) *If $\dim \text{Isom}(M) \geq 2n - 6$ then M is either homotopy equivalent to a rank one symmetric space or isometric to a homogeneous space with positive curvature.*

One of the main new tools is the so called connectedness Lemma, which turns out to be very powerful.

LEMMA (Connectedness Lemma). *If M^n has positive curvature, and N is a totally geodesic submanifold of codimension $k \leq (n+1)/2$, then the inclusion $N \hookrightarrow M$ is $n-2k+1$ connected.*

The proof is surprisingly simple and similar to the proof of Synge's theorem. One shows that in the loop space $\Omega_N(M)$ of curves starting and ending at N every critical point, i.e. geodesic starting and ending perpendicular to N , has index at least $n - 2k + 1$ since there are $n - 2k + 1$ parallel Jacobi fields starting and ending perpendicular to N . This implies

that the inclusion $N \rightarrow \Omega_N(M)$ is $n - 2k$ connected, and hence $N \rightarrow M$ is $n - 2k + 1$ connected.

An important consequence is a certain kind of periodicity in cohomology:

LEMMA (Periodicity Theorem). *If M^n has positive curvature, and N is a totally geodesic submanifold of codimension k , then there exists cohomology class $e \in H^k(M, \mathbb{Z})$ such that $\cup e : H^i(M, \mathbb{Z}) \rightarrow H^{i+k}(M, \mathbb{Z})$ is an isomorphism for $k \leq i \leq n - 2k$.*

Along the way to proving these results, he obtains a number of fundamental obstructions on the structure of the possible isotropy groups of the action. We mention a few striking examples.

THEOREM (Willing). *Let M^n be a compact simply connected manifold with positive curvature on which G acts by isometries with principal isotropy group H .*

- (a) *If H is non-trivial, then $\partial(M/G)$ is non-empty.*
- (b) *Every irreducible subrepresentation of G/H is a subrepresentation of K/H where K is an isotropy group and K/H is a sphere.*
- (c) *If $\dim(M/G) = k$, then $\partial(M/G)$ has at most $k+1$ faces, and in the case of equality M/G is homeomorphic to a simplex.*

Part (a) is powerful since the distance to the boundary is a strictly convex function. In general one can use Alexandrov geometry on the quotient as is an important tool, see the article by Fernando Galaz-Garcia in this volume. Part (b) has strong implications for the pair (G, H) , with a very short list of possibilities when the rank of H is bigger than 1.

Recently, L.Kennard proved two theorems concerning the Hopf conjectures with symmetry [Ke], see also his article in this volume.

THEOREM (Kennard). *Let M^n be a compact simply connected manifold with positive sectional curvature.*

- (a) *If $n = 4k$ and T^r acts effectively and isometrically with $r \geq 2 \log_2(n)$, then $\chi(M) > 0$.*
- (b) *Suppose M^n has the rational cohomology of a simply connected, compact symmetric space N . If T^r acts isometrically with $r \geq 2 \log_2 n + 7$, then N is a product of spheres times either a rank one symmetric space or a rank p Grassmannian $\mathrm{SO}(p+q) = \mathrm{SO}(p) \mathrm{SO}(q)$ with $p = 2$ or $p = 3$.*

The main new tool is to use the action of the Steenrod algebra to improve periodicity theorems. For example, the analogue of the connectedness Lemma is

THEOREM (Kennard). *Let M^n be a compact simply connected manifold.*

- (a) *If M^n has positive curvature and contains a pair of totally geodesic, transversely intersecting submanifolds of codimensions k_1, k_2 such that $2k_1 + 2k_2 \leq n$, then $H^*(M; \mathbb{Q})$ is $\gcd(4, k_1, k_2)$ -periodic.*
- (b) *If $H^*(M; \mathbb{Z})$ is k -periodic with $3k \leq n$, then $H^*(M; \mathbb{Q})$ is $\gcd(4, k)$ -periodic.*

Here the cohomology is called k -periodic if there exists cohomology class $e \in H^k(M, \mathbb{Z})$ such that $\cup e : H^i(M, \mathbb{Z}) \rightarrow H^{i+k}(M, \mathbb{Z})$ is an isomorphism for $0 < i < n - k$, surjective for $i = 0$ and injective for $i = n - k$. In particular, $H^{ik}(M, \mathbb{Z}) \simeq \mathbb{Z}$ for $0 \leq i \leq n - 2k - 1$.

One conclusion one can draw from these results is that positive curvature with a large isometry group can only be expected in low dimensions. This is born out in the classification of cohomogeneity one manifolds with positive curvature. We first need to describe the structure of such manifolds.

A simply connected compact cohomogeneity one manifold is the union of two homogeneous disc bundles. Given compact Lie groups H, K^-, K^+ and G with inclusions $H \subset K^\pm \subset G$ satisfying $K^\pm/H = \mathbb{S}^{\ell^\pm}$, the transitive action of K^\pm on \mathbb{S}^{ℓ^\pm} extends to a linear action on the disc $\mathbb{D}^{\ell^\pm+1}$. We can thus define

$$M = G \times_{K^-} \mathbb{D}^{\ell_-+1} \cup G \times_{K^+} \mathbb{D}^{\ell_++1}$$

glued along the boundary $\partial(G \times_{K^\pm} \mathbb{D}^{\ell^\pm+1}) = G \times_{K^\pm} K^\pm/H = G/H$ via the identity. G acts on M on each half via left action in the first component. This action has principal isotropy group H and singular isotropy groups K^\pm . One possible description of a cohomogeneity one manifold is thus simply in terms of the Lie groups $H \subset \{K^-, K^+\} \subset G$.

The simplest example is $\{e\} \subset \{S^1, S^1\} \subset S^1$ which is the manifold \mathbb{S}^2 with $G = S^1$ fixing north and south pole (and thus $K^\pm = G$) and principal isotropy trivial. The isotropy groups $\{e\} \subset \{S^1 \times \{e\}, \{e\} \times S^1\} \subset S^1 \times S^1$ describe the 3-sphere $\mathbb{S}^3 \subset \mathbb{C} \oplus \mathbb{C}$ on which $S^1 \times S^1$ acts in each coordinate. More subtle is the example $S(O(1) O(1) O(1)) \subset \{S(O(2) O(1)), S(O(1) O(2))\} \subset SO(3)$. This is the 4-sphere, thought of as the unit sphere in the set of 3×3 symmetric traceless matrices, on which $SO(3)$ acts via conjugation.

The first new family of cohomogeneity one manifolds we denote by $P_{(p_-, q_-), (p_+, q_+)}$, and is given by the group diagram

$$H = \{\pm(1, 1), \pm(i, i), \pm(j, j), \pm(k, k)\} \subset \{(e^{ip_-t}, e^{iq_-t}) \cdot H, (e^{jp_+t}, e^{jq_+t}) \cdot H\} \subset S^3 \times S^3.$$

where $\gcd(p_-, q_-) = \gcd(p_+, q_+) = 1$ and all 4 integers are congruent to 1 mod 4.

The second family $Q_{(p_-, q_-), (p_+, q_+)}$ is given by the group diagram

$$H = \{(\pm 1, \pm 1), (\pm i, \pm i)\} \subset \{(e^{ip_-t}, e^{iq_-t}) \cdot H, (e^{jp_+t}, e^{jq_+t}) \cdot H\} \subset S^3 \times S^3,$$

where $\gcd(p_-, q_-) = \gcd(p_+, q_+) = 1$, q_+ is even, and p_-, q_-, p_+ are congruent to 1 mod 4.

Special among these are the manifolds $P_k = P_{(1,1), (1+2k, 1-2k)}$, $Q_k = Q_{(1,1), (k, k+1)}$ with $k \geq 1$, and the exceptional manifold $R^7 = Q_{(-3,1), (1,2)}$. In terms of these descriptions, we can state the classification, see [Ve, GWZ].

THEOREM. (*Verdiani, n even, Grove-Wilking-Ziller, n odd*) *A simply connected cohomogeneity one manifold M^n with an invariant metric of positive sectional curvature is equivariantly diffeomorphic to one of the following:*

- *An isometric action on a rank one symmetric space,*
- *One of E_p^7, B_p^{13} or B^7 ,*
- *One of the 7-manifolds $P_k = P_{(1,1), (1+2k, 1-2k)}$, $Q_k = Q_{(1,1), (k, k+1)}$ with $k \geq 1$, or the exceptional manifold $R^7 = Q_{(-3,1), (1,2)}$*

with one of the actions described above.

Here P_k, Q_k, R should be considered as candidates for positive curvature. Recently the exceptional manifold R was excluded [VZ2]:

THEOREM (Verdiani-Ziller). *Let M be one of the 7-manifolds $Q_{(p_-, q_-), (p_+, q_+)}$ with its cohomogeneity one action by $G = S^3 \times S^3$ and assume that M is not of type $Q_k, k \geq 0$. Then there exists no analytic metric with non-negative sectional curvature invariant under G , although there exists a smooth one.*

In particular, it cannot carry an invariant metric with positive curvature.

Among the candidates P_k, Q_k the first in each sequence admit an invariant metric with positive curvature since $P_1 = S^7$ and $Q_1 = W_{1,1}^7$. The first success in the Grove program to find a new example with positive curvature is the following [GVZ, De]:

THEOREM (Grove-Verdiani-Ziller, Dearriscott). *The cohomogeneity one manifold P_2 carries an invariant metric with positive curvature.*

As for the topology of this manifold one has the following classification [Go]:

THEOREM (Goette). *The cohomogeneity one manifold P_k is diffeomorphic to $E_k \# \Sigma^{\frac{k(k+1)}{2}}$, where E_k is the S^3 principal bundle over S^4 with Euler class k , and Σ is the Gromoll-Meyer generator in the group of exotic 7-spheres.*

In particular, it follows that P_2 is homeomorphic but not diffeomorphic to $T_1 S^4$, and thus indeed a new example of positive curvature.

4. FIBRATIONS WITH POSITIVE CURVATURE

As we saw in Section 2, many of the examples of positive curvature are the total space of a fibration. It is thus natural to ask under what condition the total space admits positive curvature, if the base and fiber do. One should certainly expect conditions, since the bundle could be trivial.

A. Weinstein examined this question in the context of Riemannian submersions with totally geodesic fibers [We]. He called a bundle fat if $\sec(X, U) > 0$ for all 2-planes spanned by a vector U tangent to the fibers and X orthogonal to the fibers. For simplicity, we restrict ourselves for the moment to principle bundles. Let $\pi: P \rightarrow B$ be a G -principle bundle. Given a metric on the base $\langle \cdot, \cdot \rangle_B$, a principal connection $\theta: TP \rightarrow \mathfrak{g}$, and a fixed biinvariant metric Q on G , one defines a Kaluza Klein metric on P as:

$$g_t(X, Y) = tQ(\theta(X), \theta(Y)) + g(\pi_*(X), \pi_*(Y)).$$

Here one has the additional freedom to modify t , in fact $t \rightarrow 0$ usually increases the curvature.

The projection π is then a Riemannian submersion with totally geodesic fibers isometric to (G, tQ) . Weinstein observed that the fatness condition (for any t) is equivalent to requiring that the curvature Ω of θ has the property that $\Omega_u = Q(\Omega, u)$ is a symplectic 2-form on the horizontal space, i.e. the vector space orthogonal to the fibers, for every $u \in \mathfrak{g}$. If $G = S^1$, this is equivalent to the base being symplectic. Fatness of the principal

connection is already a strong condition on the principal bundle which one can express in terms of the characteristic classes of the bundle.

A. Weinstein made the following observation. Assume that G and B^{2n} are compact and connected. For each $y \in \mathfrak{g}$, we have a polynomial $q_y : \mathfrak{g} \rightarrow \mathbb{R}$ given by

$$q_y(\alpha) = \int_G \langle Ad_g(y), \alpha \rangle^{2n} dg$$

which one easily checks is Ad_G -invariant (in α and y). By Chern–Weyl theory, there exists a closed $2n$ -form ω_y on B^{2n} such that $\tau^*\omega_y = q_y(\Omega)$ and $[\omega_y] \in H^{2n}(B, \mathbb{R})$ represents a characteristic class of the bundle. If the bundle is fat, $\Omega_y^{2n} \neq 0$ is a volume form on H . Thus, if G is connected, $\langle Ad_g(y), \Omega \rangle^{2n}$ is nowhere zero and has constant sign when g varies along G , and the integral is thus $q_y(\Omega)$ nonzero on H . Hence ω_y is a volume form of B^{2n} , in particular B^{2n} is orientable, and the characteristic number $\int_B \omega_y$ is nonzero. In [FZ3] this characteristic number was called the Weinstein invariant associated to $y \in \mathfrak{t}$ and was computed explicitly in terms of Chern and Pontrjagin numbers. For each adjoint orbit in \mathfrak{t} one obtains obstructions to fatness in terms of Chern and Pontryagin numbers. The above discussion easily generalizes to fiber bundles associated to principle bundles to obtain obstructions. We call the metric on the fiber bundle a connection metric if the fibers are totally geodesic. Some sample obstructions are:

- THEOREM. (a) ([DR]) *The only \mathbb{S}^3 bundle over \mathbb{S}^4 that admits a fat connection metric is the Hopf bundle $\mathbb{S}^3 \rightarrow \mathbb{S}^7 \rightarrow \mathbb{S}^4$.*
- (b) [FZ3] *The only two \mathbb{S}^3 sphere bundles over \mathbb{CP}^2 that may admit a fat connection metric are the complex sphere bundles with $c_1^2 = 9$ and $c_2 = 1$ or 2 . In particular, $T_1\mathbb{CP}^2 \rightarrow \mathbb{CP}^2$ does not have a fat connection metric.*
- (c) [FZ3] *If a sphere bundle over B^{2n} admits a fat connection metric, then the Pontryagin numbers satisfy $\det(p_{j-i+1})_{1 \leq i, j \leq n} \neq 0$.*

On the other hand, many of the examples in Section 2 are fat fiber bundles, and the new example of positive curvature in Section 3 is a fat bundle as well.

If one wants to achieve positive curvature on the total space, we need to assume, in addition to the base having positive curvature, that $G = \mathbb{S}^1$, $\mathrm{SU}(2)$ or $\mathrm{SO}(3)$. In [CDR] a necessary and sufficient condition for positive curvature of such metrics was given. The proof carries over immediately to the category of orbifold principal bundles, which includes the case where the G action on P has only finite isotropy groups.

THEOREM 4.1. (Chaves-Derdzinski-Rigas) *A connection metric g_t on an orbifold G -principal bundle with $\dim G \leq 3$ has positive curvature, for t sufficiently small, if and only if*

$$(\nabla_x \Omega_u)(x, y)^2 < |i_x \Omega_u|^2 k_B(x, y),$$

for all linearly independent horizontal vectors x, y and $0 \neq u \in \mathfrak{g}$.

Here $k_B(x, y) = g(R_B(x, y)y, x)$ is the unnormalized sectional curvature and $i_x \Omega_u \neq 0$ is precisely the above fatness condition.

All the known examples of principle bundles with positive curvature satisfy this condition (called hyperfatness in [Zi2]). The new example in Section 3 also is such an $G = \mathbb{S}^3$ principle

bundle, if one allows the action of G to be almost free (the base is then an orbifold). All of the above easily carries over to orbifold principle bundles as well.

The condition is of course trivially satisfied if $\nabla\Omega = 0$. This is equivalent to the metric on B to be quaternionic Kähler. Thus a quaternionic Kähler manifold with positive sectional curvature gives rise to a positively curved metric on the S^3 or $SO(3)$ principal bundle defined canonically by its structure. Unfortunately, if $\dim > 4$, a positively curved quaternionic Kähler metric is isometric to $\mathbb{H}\mathbb{P}^n$ only gives rise to $S^{4n+3}(1)$ on the total space of the principle bundle (Berger). And in dimension 4, the only smooth quaternionic Kähler metric (with positive scalar curvature) is isometric to $S^4(1)$ and $\mathbb{C}\mathbb{P}^2$ giving rise to $S^7(1)$ and the positively curved Aloff Wallach space $W_{1,1}$. Thus in the smooth category, it gives nothing new.

But if the quotient is a quaternionic Kähler orbifold, there are many examples. In the case of the candidate P_k , the subgroup $S^3 \times \{e\} \subset S^3 \times S^3$ acts almost freely and the quotient is an orbifold homeomorphic to S^4 . Similarly, for Q_k , the subgroup $S^3 \times \{e\} \subset S^3 \times S^3$ acts almost freely with quotient an orbifold homeomorphic to $\mathbb{C}\mathbb{P}^2$. On these two orbifolds, Hitchin [Hi] constructed a quaternionic Kähler metric with positive scalar curvature, and in [GWZ] it was shown that the total space of the canonically defined principal S^3 resp. $SO(3)$ principal bundle is equivariantly diffeomorphic to P_k resp. Q_k . In [Zi3] it was shown that the Hitchin metrics have a large open set on which the curvature is positive, but not quite everywhere. Nevertheless, in [GWZ] and [De] this Hitchin metric was the starting point. In [GWZ] a connection metric was constructed on P_2 by defining the metric piecewise via low degree polynomials, both for the metric on the base, and the principle connection. The metric is indeed very close to the Hitchin metric. In [De] the quaternionic Kähler Hitchin metric on S^4 was deformed, but the the principle connection stayed the same. The metric on S^4 was then approximated by polynomials in order to show the new metric has positive curvature.

In both cases, the proof that the polynomial metric has positive curvature, was carried out by using a method due to Thorpe (and a small modification of it in [De]).

We finish by describing this Thorpe method since it is not so well known, but very powerful (see [Th1], [Th2] and also [Pü]). In fact, one of the problems of finding new examples is that after constructing a metric, showing that it has positive curvature is difficult. Even the linear algebra problem for a curvature tensor on a vector space is highly non-trivial! Here is where the Thorpe method helps.

Let V be a vector space with an inner product and R a 3-1 tensor which satisfies the usual symmetry properties of a curvature tensor. We can regard R as a linear map

$$\hat{R}: \Lambda^2 V \rightarrow \Lambda^2 V,$$

which, with respect to the natural induced inner product on $\Lambda^2 V$, becomes a symmetric endomorphism. The sectional curvature is then given by:

$$\sec(v, w) = \langle \hat{R}(v \wedge w), v \wedge w \rangle$$

if v, w is an orthonormal basis of the 2-plane they span.

If \hat{R} is positive definite, the sectional curvature is clearly positive as well. But this condition is exceedingly strong since a manifold with $\hat{R} > 0$ is covered by a sphere [BW]. But one can modify the curvature operator by using a 4-form $\eta \in \Lambda^4(V)$. It induces

another symmetric endomorphism $\hat{\eta}: \Lambda^2 V \rightarrow \Lambda^2 V$ via $\langle \hat{\eta}(x \wedge y), z \wedge w \rangle = \eta(x, y, z, w)$. We can then consider the modified curvature operator $\hat{R}_\eta = \hat{R} + \hat{\eta}$. It satisfies all symmetries of a curvature tensor, except for the Bianchi identity. Clearly \hat{R} and \hat{R}_η have the same sectional curvature since

$$\langle \hat{R}_\eta(v \wedge w), v \wedge w \rangle = \langle \hat{R}(v \wedge w), v \wedge w \rangle + \eta(v, w, v, w) = \sec(v, w)$$

If we can thus find a 4-form η with $\hat{R}_\eta > 0$, the sectional curvature is positive. Thorpe showed [Th2] that in dimension 4, one can always find a 4-form such that the smallest eigenvalue of \hat{R}_η is also the minimum of the sectional curvature, and similarly a possibly different 4-form such that the largest eigenvalue of \hat{R}_η is the maximum of the sectional curvature. Indeed, if an eigenvector ω to the largest eigenvalue of \hat{R} is decomposable, the eigenvalue is clearly a sectional curvature. If it is not decomposable, then $\omega \wedge \omega \neq 0$ and one easily sees that \hat{R}_η with $\eta = \omega \wedge \omega$ has a larger eigenvalue.

This is not the case anymore in dimension bigger than 4 [Zo]. Nevertheless this can be an efficient method to estimate the sectional curvature of a metric. In fact, Püttmann [Pü] used this to compute the maximum and minimum of the sectional curvature of all positively curved homogeneous spaces, which are not spheres. It is peculiar to note though that this method does not work to determine which homogeneous metrics on \mathbb{S}^7 have positive curvature, see [VZ1].

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